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Relativistic generalisation of the Kroll–Watson formula†

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Abstract. The relativistic analogue of the space-translation method is derived. Using this method the generalisation of the Kroll-Watson formula is obtained for the scattering of an arbitrary charged particle (e.g., mesons, hyperons, quarks, etc). The separation of the background and resonant parts of the scattering amplitude is predicted.

1. Introduction

In the last decade a surge of interest in atomic physics with a laser field has been observed. The applications of lasers to many branches of applied physics are the reasons for this trend (see, e.g., Bunkin *et al* 1972, Grey Morgan 1975). The electron-atom scattering within the laser field is also interesting because of the possible application of this process to the measurement of off-shell scattering matrix elements which cannot be observed without the use of an electromagnetic field. The appearance of the off-shell elements is due to the role of photon as the 'third body', which can be absorbed or emitted in the course of scattering. This fact has been well known for almost fifty years (Nordsick 1937) and some attempts have been made to use the bremsstrahlung process for measurements of the off-shell scattering matrix elements in nuclear physics. Because the intensity and frequency of the laser can be more precisely controlled, therefore, it is hoped that the applications of lasers in replacing bremsstrahlung radiation can appear to be more profound.

The use of lasers in scattering processes introduces additional parameters: frequency, intensity, polarisation, the structure of modes, etc. Among these parameters the frequency and intensity seem to be important and the problem has been investigated only for the following regions of these parameters: (i) low intensities and arbitrary frequencies (perturbation theory calculations (for a review see Gavrila and van der Wiel 1978)); (ii) arbitrary (but not very high) intensities and low frequencies (Kroll and Watson 1973; for a review see also Mittleman 1982b, Rosenberg 1982); (iii) high intensities and high frequencies (Gavrila and Kamiński 1984).

In quantum electrodynamics the laser field is described as the quantum state of the electromagnetic field. For instance, the single-mode laser is best described by a quantum coherent state with a stochastic phase. It appears, however, that in cases of strong laser fields the classical description becomes correct (Białynicka-Birula and Białynicka-Birula 1976, Mittleman 1982b). Since we will deal with sufficiently powerful lasers, therefore, the classical description of the laser field can be adopted here.

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Moreover, the laser field will be considered as a linearly polarised monochromatic plane wave. The linear polarisation is for simplicity and the possible application of the result obtained to the resonant scattering, but the monochromaticity of the field has to be justified. The laser field in many cases consists of a collection of closely spaced modes such that $\Delta\omega/\omega \ll 1$, where ω is the central frequency and $\Delta\omega$ is the mode spacing. In these cases the electromagnetic field can be considered as a wave of frequency ω slowly varying in time amplitude and phase. This means that any calculation carried out for a single-mode laser field can be adapted to the multimode case by taking the cross section calculated with the monochromatic electromagnetic field, and averaging it over the adiabatic variations in time of the laser amplitude and phase. The calculations along this path has been started to be carried out† (Bunkin and Karapetyan 1971, Krüger and Jung 1978, Jung 1980, Zoller 1980, Daniele and Ferrante 1982, Daniele *et al* 1983). Such an approach, however, accounts only for the pulse shape effects and does not allow a full description of the influence of realistic high power multimode lasers on the processes considered. We will not deal with these problems in this paper.

It is well known from perturbation theory calculations that the interaction of a charged particle with a low-frequency radiation field modifies the scattering process to such an extent that the scattering amplitude is equal to that without the radiation field times the frequency-dependent factor. This is the so-called Low theorem (Low 1958). Bunkin and Fedorov (1965) obtained a similar factorisation by considering potential scattering in the Born approximation within a monochromatic laser field. Kroll and Watson (1973) combined both these results showing that for potential scattering within the low-frequency laser field the following factorisation of the differential cross section $\mathrm{d}\sigma^n/\mathrm{d}\Omega$ for an electron scattering from the initial state with momentum $\bar{p}_\mathrm{i}$ and energy $E_\mathrm{i}=\bar{p}_\mathrm{i}^2/2m$ into a final state with momentum $\bar{p}_\mathrm{f}$ and energy $E_\mathrm{f}=\bar{p}_\mathrm{f}^2/2m = E_\mathrm{i}+n\omega$, occurs‡

$$\frac{\mathrm{d}\sigma^n}{\mathrm{d}\Omega}=\frac{p_\mathrm{f}}{p_\mathrm{i}}J_n^2(\bar{\alpha}_0(\bar{p}_\mathrm{f}-\bar{p}_\mathrm{i}))\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\bar{p}_\mathrm{f},\bar{p}_\mathrm{i})\bigg|_{\text{no laser}}+\mathrm{O}\left(\frac{\omega}{E_\mathrm{i}}\right), \tag{1}$$

where $\bar{\alpha}_0=-e\bar{a}/m\omega$ and the vector potential $\bar{A}(t)$ in the dipole approximation was assumed to be

$$\bar{A}(t)=\bar{a}\cos(\omega t). \tag{2}$$

J_n is the Bessel function of order n. It also appears that the corrections which are linear in ω can be incorporated into the first term in (1) leading to

$$\frac{\mathrm{d}\sigma^n}{\mathrm{d}\Omega}=\frac{p_\mathrm{f}}{p_\mathrm{i}}J_n^2(\bar{\alpha}_0(\bar{p}_\mathrm{f}-\bar{p}_\mathrm{i}))\frac{\mathrm{d}\sigma^\mathrm{el}}{\mathrm{d}\Omega}(\bar{Q}_\mathrm{f},\bar{Q}_\mathrm{i})+\mathrm{O}\left(\frac{\omega^2}{E_\mathrm{i}^2}\right), \tag{3}$$

where $\mathrm{d}\sigma^\mathrm{el}/\mathrm{d}\Omega$ is the elastic differential cross section for an electron scattering from the initial state with momentum $\bar{Q}_\mathrm{i}$ into a final state with momentum $\bar{Q}_\mathrm{f}$ without the radiation field and

$$\bar{Q}_{\mathrm{i,f}}=\bar{p}_{\mathrm{i,f}}-nm\omega\bar{\alpha}_0/\bar{\alpha}_0(\bar{p}_\mathrm{f}-\bar{p}_\mathrm{i}).$$

† This kind of investigation is at its very beginning in the atomic collision theory and is based on results which have already been achieved in the related field of the interaction of a strong radiation field with isolated atoms and molecules.

‡ The units in which $\hbar=c=1$ and the fine-structure constant $\alpha=e^2/4\pi$ are used.

It can be checked that $\bar{Q}_i^2 = \bar{Q}_f^2$, i.e., the off-shell scattering matrix elements are proportional to ω^2/E_i^2 (Mittleman 1979). The parameter α_0 can be interpreted as the amplitude of the classical motion of the electron within the monochromatic plane wave. Therefore, it is obvious that it cannot be arbitrarily large if one applies non-relativistic quantum mechanics. The discussion of this problem can be found in Drühl and McIver (1983).

The aim of this paper is to consider the relativistic generalisation of the Kroll-Watson formula. The subject seems to be interesting in the context of resonant scattering when the background does not allow one to carry out the precise measurement of both the resonant energy and width. The important paper of Jung and Taylor (1981) proposed exploiting the laser to suppress background terms in the scattering amplitude and thus study only the rapidly varying parts. In this paper I argue that such a phenomenon can occur also in relativistic scattering.

Since the background is assumed to be described by a static potential, therefore, the potential scattering of charged particles in the laser field is considered. The laser field, as has been discussed previously, can be described by the monochromatic plane wave of infinite extent, namely, $A_{R\mu}(kx) = a_\mu \cos(kx)$. Moreover, the Lorentz gauge for the radiation field is followed by $ak = 0$.

The plan of this paper is as follows. Section 2 contains a discussion of the Volkov solution (Volkov 1935, 1937). The scattering of electrons by the static spin-independent potential $\mathscr{A}_\mu(\bar{x})$ in the Born approximation within the plane wave is studied in § 3. In § 4 the relativistic generalisation of the Kroll-Watson formula to the case of the static spin-dependent potential and low-frequency laser field is derived. Section 5 contains the discussion.

2. Volkov solution

The relativistic electron in the monochromatic plane wave is described by the Volkov solution of the Dirac equation, namely†

$$\psi[x; p, \lambda | A_R] = \left(1 + \frac{e}{2kp} \not{k} \not{A}_R(kx)\right)$$
$$\times \exp\left[-ipx - i\int_0^{kx} \left(\frac{e}{kp} pA_R(\theta) - \frac{e^2}{2kp} A_R^2(\theta)\right) d\theta\right] u(p, \lambda), \quad (4)$$

where p and λ are the electron momentum and polarisation, respectively. I adopt the normalisation

$$u^+(p, \lambda) u(p, \lambda) = E/m, \quad (5)$$

where $E = p_0$. With this condition the number of all possible final states in the momentum volume $d\bar{p}_f$ and the incident flux are equal to $d\bar{p}_f/(2\pi)^3$ and $|\bar{p}_i|/E_i$, respectively. For simplicity, the summation over the final electron polarisation and the average over the incident electron polarisation will be carried out with the help of the formula

$$\sum_\lambda u(p, \lambda) \bar{u}(p, \lambda) = (\not{p} + m)/2m. \quad (6)$$

† The Feynman notation $\not{a} = a_\mu \gamma^\mu$ is used.

Let me introduce, after Nikishov and Ritus (1964a, b), the family of functions $A_{j,n}$†

$$A_{j,n}(a, b)=\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta(\cos\theta)^j \exp(\mathrm{i}n\theta-\mathrm{i}a\sin\theta+\mathrm{i}b\sin 2\theta). \tag{7}$$

It is straightforward to show that

$$\sum_{e=-\infty}^{\infty} A_{j,e}(a_1, b_1)A^{*}_{j',e-n}(a_2, b_2)=A_{j+j',n}(a_1-a_2, b_1-b_2). \tag{8}$$

With the help of these functions the Volkov solution can be written in the form

$$\begin{aligned}\psi[x; p, \lambda|A_\mathrm{R}] &= \sum_{n=-\infty}^{\infty}\left[A_{0,n}\left(\frac{eap}{kp}, \frac{e^2a^2}{8kp}\right)+\frac{e}{2kp}\not{K}\not{a}A_{1,n}\left(\frac{eap}{kp}, \frac{e^2a^2}{8kp}\right)\right]\\ &\quad\times\exp\left[-\mathrm{i}\left(p+\frac{e^2a^2}{4kp}k+nk\right)x\right]u(p, \lambda).\end{aligned} \tag{9}$$

3. Cross section in the Born approximation

The scattering matrix of the process considered is equal to

$$\begin{aligned}S_\mathrm{fi} &= S[p_\mathrm{f}, \lambda_\mathrm{f}; p_\mathrm{i}, \lambda_\mathrm{i}|A_\mathrm{R}, \mathscr{A}]\\ &= -\mathrm{i}m(E_\mathrm{i}E_\mathrm{f})^{-1/2}\int \mathrm{d}x\,\bar{\psi}[x; p_\mathrm{f}, \lambda_\mathrm{f}|A_\mathrm{R}]e\not{\mathscr{A}}(\bar{x})\psi[x; p_\mathrm{i}, \lambda_\mathrm{i}|A_\mathrm{R}, \mathscr{A}]\end{aligned} \tag{10}$$

where $\psi[x; p_\mathrm{i}, \lambda_\mathrm{i}|A_\mathrm{R}, \mathscr{A}]$ is the exact solution of the Dirac equation,

$$(\mathrm{i}\gamma^\mu\partial_\mu - e\not{\mathscr{A}}(\bar{x}) - e\not{A}_\mathrm{R}(kx) - m)\psi = 0, \tag{11}$$

which fulfils the initial condition,

$$\psi[x; p_\mathrm{i}, \lambda_\mathrm{i}|A_\mathrm{R}, \mathscr{A}] \xrightarrow[t\to-\infty]{} \psi[x; p_\mathrm{i}, \lambda_\mathrm{i}|A_\mathrm{R}]. \tag{12}$$

For the external potential which is independent of time the scattering matrix can be put down in the following form:

$$S_\mathrm{fi} = -2\pi\mathrm{i}\sum_{n=-\infty}^{\infty}\delta\left[E_\mathrm{f}-E_\mathrm{i}-n\omega+\frac{e^2a^2\omega}{4}\left(\frac{1}{kp_\mathrm{f}}-\frac{1}{kp_\mathrm{i}}\right)\right]T^{(n)}_\mathrm{fi}. \tag{13}$$

The transition amplitude $T^{(n)}_\mathrm{fi}$ in the Born approximation is equal to

$$\begin{aligned}T^{(n)}_\mathrm{fi,B} &= em(E_\mathrm{i}E_\mathrm{f})^{-1/2}\tilde{\mathscr{A}}_\mu\left[\bar{p}_\mathrm{f}-\bar{p}_\mathrm{i}-n\bar{k}+\frac{e^2a^2}{4}\left(\frac{1}{kp_\mathrm{f}}-\frac{1}{kp_\mathrm{i}}\right)\bar{k}\right]\\ &\quad\times\left[A_{0,n}\bar{u}(p_\mathrm{f}, \lambda_\mathrm{f})\gamma^\mu u(p_\mathrm{i}, \lambda_\mathrm{i})+A_{1,n}\left(\frac{e}{2kp_\mathrm{f}}\bar{u}(p_\mathrm{f}, \lambda_\mathrm{f})\not{a}\not{K}\gamma^\mu u(p_\mathrm{i}, \lambda_\mathrm{i})\right.\right.\\ &\quad\left.+\frac{e}{2kp_\mathrm{i}}\bar{u}(p_\mathrm{f}, \lambda_\mathrm{f})\gamma^\mu\not{K}\not{a}u(p_\mathrm{i}, \lambda_\mathrm{i})\right)\\ &\quad\left.+A_{2,n}\frac{e^2}{4(kp_\mathrm{f})(kp_\mathrm{i})}\bar{u}(p_\mathrm{f}, \lambda_\mathrm{f})\not{a}\not{K}\gamma^\mu\not{K}\not{a}u(p_\mathrm{i}, \lambda_\mathrm{i})\right],\end{aligned} \tag{14}$$

† All these functions can be expressed in terms of the so-called generalised Bessel functions, properties of which were studied by Leubner (1981).

where

$$A_{j,n}=A_{j,n}\left(\frac{eap_i}{kp_i}-\frac{eap_f}{kp_f},\frac{e^2a^2}{8kp_i}-\frac{e^2a^2}{8kp_f}\right), \tag{15}$$

and the Fourier transform of the potential

$$\tilde{\mathscr{A}}_\mu(\bar{q})=\int \mathrm{d}\bar{x}\exp(-\mathrm{i}\bar{x}\bar{q})\mathscr{A}_\mu(\bar{x}). \tag{16}$$

Summing over the initial and averaging over the final polarisations the following expression for the differential cross section in the Born approximation can be obtained:

$$\begin{aligned}\frac{\mathrm{d}\sigma_\mathrm{B}}{\mathrm{d}\Omega}&=\sum_{n=-\infty}^{\infty}\frac{\mathrm{d}\sigma^{(n)}_{\mathrm{fi,B}}}{\mathrm{d}\Omega}\\&=\sum_{n=-\infty}^{\infty}\int \mathrm{d}E_f\frac{|\bar{p}_f|}{|\bar{p}_i|}\frac{E_iE_f}{4\pi^2}\\&\quad\times\delta\left[E_f-E_i-n\omega+\frac{e^2a^2\omega}{4}\left(\frac{1}{kp_f}-\frac{1}{kp_i}\right)\right]|T^{(n)}_{\mathrm{fi,B}}|^2_{\mathrm{AV}},\end{aligned} \tag{17}$$

where

$$\begin{aligned}|T^{(n)}_{\mathrm{fi,B}}|^2_{\mathrm{AV}}&=\frac{1}{2}\sum_{\lambda_i,\lambda_f}|T^{(n)}_{\mathrm{fi,B}}|^2\\&=\frac{e^2}{8E_iE_f}\tilde{\mathscr{A}}_\mu\left[\bar{p}_f-\bar{p}_i-n\bar{k}+\frac{e^2a^2}{4}\left(\frac{1}{kp_f}-\frac{1}{kp_i}\right)\bar{k}\right]\\&\quad\times\tilde{\mathscr{A}}_\nu\left[\bar{p}_f-\bar{p}_i-n\bar{k}+\frac{e^2a^2}{4}\left(\frac{1}{kp_f}-\frac{1}{kp_i}\right)\bar{k}\right]\\&\quad\times\sum_{i=1,2}\sum_{j_1,j_2,j_3,j_4=1,2}A_{j_1+j_2,n}A^*_{j_3+j_4,n}T^{\mu\nu}(i;j_1,j_2,j_3,j_4),\end{aligned} \tag{18}$$

and

$$\begin{aligned}T^{\mu\nu}&(1;j_1,j_2,j_3,j_4)\\&=\left(\frac{e}{2kp_i}\right)^{j_2+j_3}\left(\frac{e}{2kp_f}\right)^{j_1+j_4}\\&\quad\times\mathrm{Tr}[\not{p}_f(\not{a}\not{K})^{j_1}\gamma^\mu(\not{K}\not{a})^{j_2}\not{p}_i(\not{a}\not{K})^{j_3}\gamma^\nu(\not{K}\not{a})^{j_4}],\\T^{\mu\nu}&(0;j_1,j_2,j_3,j_4)\\&=m^2\left(\frac{e}{2kp_i}\right)^{j_2+j_3}\left(\frac{e}{2kp_f}\right)^{j_1+j_4}\\&\quad\times\mathrm{Tr}[(\not{a}\not{K})^{j_1}\gamma^\mu(\not{K}\not{a})^{j_2}(\not{a}\not{K})^{j_3}\gamma^\nu(\not{K}\not{a})^{j_4}].\end{aligned}$$

In the formula (18) the abbreviation (15) has been used.

The final expression for the differential cross section in the Born approximation is rather complicated. Nevertheless, the formulae (17) and (18) can be simplified under the spatial conditions. In the following I will consider the case in which the angles between $\bar{p}_f$ and $\bar{p}_i$, and the polarisation vector of the laser field $\bar{a}$ are different from zero. Let me denote by E the energy of the incident or scattered particle. I will assume further that the quantity

$$\varkappa = (I/I_0)^{1/2}(m/\omega)^2|\bar{p}|/E \tag{19}$$

is of the order of unity. In equation (19) m is the electron mass, I the laser intensity in $\mathrm{W\,cm^{-2}}$ and $I_0 = 4.7\times10^{29}\ \mathrm{W\,cm^{-2}}$. Moreover, the laser frequency ω is assumed to be much less than the energy E, i.e.,

$$\omega/E \ll 1. \tag{20}$$

Under these assumptions the cross section for n-photon absorption in the Born approximation is of the form

$$\frac{\mathrm{d}\sigma_{\mathrm{B}}^{(n)}}{\mathrm{d}\Omega} = \left| A_{0,n}\left(\frac{eap_{\mathrm{i}}}{kp_{\mathrm{i}}} - \frac{eap_{\mathrm{f}}}{kp_{\mathrm{f}}}, \frac{e^2a^2}{8kp_{\mathrm{i}}} - \frac{e^2a^2}{8kp_{\mathrm{f}}}\right)\right|^2 \frac{\mathrm{d}\sigma_{\mathrm{B}}}{\mathrm{d}\Omega}\bigg|_{\text{no laser}} + \mathrm{O}\left(\frac{\omega}{E_{\mathrm{i}}}\right), \tag{21}$$

where the final energy is determined by the δ function in (17). This is the relativistic generalisation of the Kroll–Watson formula in the Born approximation. It has to be emphasised here that the corrections to (20) are of the order of ω/E_{i}. This property of (20) is due to the Born approximation, and in the next section it will be shown that in place of (20) the condition

$$\omega/(E-m) \ll 1, \tag{22}$$

should be adopted (as in the non-relativistic case) which is followed of course by (20).

The scattering of relativistic electrons by the Coulomb potential in the presence of a monochromatic plane wave of an arbitrary polarisation has been studied previously by Denisov and Fedorov (1967). Their result for the linear polarisation agrees with (17) for the potential of the form $\mathscr{A}_\mu(\bar{x}) = V(\bar{x})g_{\mu0}$.

The calculation presented in this section seems to be helpful in obtaining the generalisation to an arbitrary potential, which is the subject of the next section.

4. Relativistic Kroll–Watson formula

In order to obtain the relativistic generalisation of the Kroll–Watson formula without the Born approximation let me consider the following Dirac equation:

$$(\mathrm{i}\gamma^\mu\partial_\mu - e\not{A}_{\mathrm{R}}(kx) - m)\psi = \mathscr{A}(\bar{x})\psi,$$

where $\mathscr{A}(\bar{x})$ is an arbitrary static potential, which can depend on any quantum numbers, i.e., spin, isospin, colour, flavour, etc. I will look for the solution of this equation which fulfils condition (12). Let me then define the new wavefunction $\Psi[x; p_{\mathrm{i}}, \lambda_{\mathrm{i}}]$ with the help of the following unitary transformation:

$$\begin{aligned}\psi[x; p_{\mathrm{i}}, \lambda_{\mathrm{i}}|A_{\mathrm{R}}, \mathscr{A}] &= \left(1 - \frac{\mathrm{i}e}{2}(k\cdot\partial)^{-1}\not{k}\not{A}_{\mathrm{R}}\right)\\ &\quad\times\exp\left(-\mathrm{i}\int_0^{kx}(e(k\cdot\partial)^{-1}(A_{\mathrm{R}}(\theta)\cdot\partial) + \frac{\mathrm{i}e^2}{2}A_{\mathrm{R}}^2(\theta)(k\partial)^{-1})\,\mathrm{d}\theta\right)\Psi[x; p_{\mathrm{i}}, \lambda_{\mathrm{i}}].\end{aligned} \tag{23}$$

It is not difficult to check that in the case of the linearly polarised laser field the modified Dirac equation for $\Psi[x; p_i, \lambda_i]$ is of the form,

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\Psi = \mathscr{A}_{\mathrm{eff}}[x|A_R, \mathscr{A}]\Psi, \tag{24}$$

where the so-called effective potential $\mathscr{A}_{\mathrm{eff}}$ is equal to

$$\begin{aligned}\mathscr{A}_{\mathrm{eff}}[x|A_R, \mathscr{A}]&\\ &= \sum_{n,n'} \exp[\mathrm{i}(n-n')x]\left[A^*_{0,n}\left(\frac{ea\hat{p}}{k\hat{p}}, \frac{e^2a^2}{8k\hat{p}}\right)\right.\\ &\quad \left.-\frac{e}{2k\hat{p}}\not{K}\not{a}A^*_{1,n}\left(\frac{ea\hat{p}}{k\hat{p}}, \frac{e^2a^2}{8k\hat{p}}\right)\right]\exp\left(\mathrm{i}\frac{e^2a^2}{4k\hat{p}}kx\right)\mathscr{A}(\bar{x})\\ &\quad \times\exp\left(-\mathrm{i}\frac{e^2a^2}{4k\hat{p}}kx\right)\left[A_{0,n'}\left(\frac{ea\hat{p}}{k\hat{p}}, \frac{e^2a^2}{8k\hat{p}}\right)\right.\\ &\quad \left.+\frac{e}{2k\hat{p}}\not{K}\not{a}A_{1,n'}\left(\frac{ea\hat{p}}{k\hat{p}}, \frac{e^2a^2}{8k\hat{p}}\right)\right],\end{aligned} \tag{25}$$

and the operator $\hat{p} = \mathrm{i}\partial$. The approach leading to (24) is the relativistic generalisation of the well known non-relativistic space-translation method (Henneberger 1968). The only difference consists in the character of the effective potential $\mathscr{A}_{\mathrm{eff}}$, which in the relativistic theory is the non-local operator-valued potential.

For further considerations it is better to write down the equation (24) in terms of the modified Fourier transform of Ψ,

$$\Psi[x; p_i, \lambda_i] = \sum_l \int \frac{\mathrm{d}\bar{q}}{(2\pi)^3} \exp\left[-\mathrm{i}\left(q + lk - \frac{e^2a^2}{4kq}k\right)x\right]\tilde{\Psi}_l[q; p_i, \lambda_i], \tag{26}$$

where $q = (E_i, \bar{q})$ and E_i is the zero component of p_i. This leads to the following equation for $\tilde{\Psi}_e$

$$\begin{aligned}\left(\not{q} + l\not{K} - \frac{e^2a^2}{4kq}\not{K} - m\right)\tilde{\Psi}_l[q; p_i, \lambda_i]&\\ &= \sum_{l'} \int \frac{\mathrm{d}\bar{q}'}{(2\pi)^3} T_{l-l'}[q, q']\tilde{\Psi}_{l'}[q'; p_i, \lambda_i],\end{aligned} \tag{27}$$

where

$$\begin{aligned}T_n[q, q'] &= A_{0,n}\tilde{\mathscr{A}}(\bar{q}-\bar{q}') + A_{1,n}\left(\frac{e\not{a}\not{K}}{2kq}\tilde{\mathscr{A}}(\bar{q}-\bar{q}') + \tilde{\mathscr{A}}(\bar{q}-\bar{q}')\frac{e\not{K}\not{a}}{2kq'}\right)\\ &\quad + A_{2,n}\frac{e^2}{4(kq)(kq')}\not{a}\not{K}\tilde{\mathscr{A}}(\bar{q}-\bar{q}')\not{K}\not{a},\end{aligned} \tag{28}$$

and the functions $A_{j,n}$ depend on $eaq'/(kq') - eaq/(kq)$ and $e^2a^2/(8kq') - e^2a^2/(8kq)$. Moreover, the four-vector q' is defined similarly to q, i.e., $q' = (E_i, \bar{q}')$. In the low-frequency limit (equation (22) with $eap/(kp)$ and $e^2a^2/(kp)$ fixed) one can keep only the first term in (28) and neglect $(l - e^2a^2/(4kq))\not{K}$ with respect to $\not{q} - m$. This brings us to a simple equation for $\tilde{\Psi}_l$, namely,

$$\begin{aligned}(\gamma q - m)\tilde{\Psi}_l[q; p_i, \lambda_i]&\\ &= \sum_{l'} \int \frac{\mathrm{d}\bar{q}}{(2\pi)^3}\tilde{\mathscr{A}}(\bar{q}-\bar{q}')A_{0,l-l'}\left(\frac{eaq'}{kq'} - \frac{eaq}{kq}, \frac{e^2a^2}{8kq'} - \frac{e^2a^2}{8kq'}\right)\tilde{\Psi}_{l'}[q'; p_i, \lambda_i].\end{aligned} \tag{29}$$

This equation can be easily solved with the result

$$\tilde{\Psi}_l[q;p_{\rm i},\lambda_{\rm i}]=A_{0,l}\left(\frac{eap_{\rm i}}{kp_{\rm i}}-\frac{eaq}{kq},\frac{e^2a^2}{8kp_{\rm i}}-\frac{e^2a^2}{8kq}\right)\tilde{\psi}[\bar{q};p_{\rm i},\lambda_{\rm i}], \tag{30}$$

where $\tilde{\psi}[\bar{q};p_{\rm i},\lambda_{\rm i}]$ is the Fourier transform of the solution of the stationary Dirac equation,

$$(\gamma^0E_{\rm i}+{\rm i}\bar{\gamma}\bar{\nabla}+m)\psi[\bar{x};p_{\rm i},\lambda_{\rm i}]=\mathscr{A}(\bar{x})\psi[\bar{x};p_{\rm i},\lambda_{\rm i}] \tag{31}$$

with the incident plane wave and the outgoing spherical wave boundary condition.

Going back to the scattering problem it is straightforward to prove that the scattering matrix of the process considered is equal to

$$S_{\rm fi}=-{\rm i}m(E_{\rm i}E_{\rm f})^{-1/2}\int{\rm d}x\,\exp({\rm i}p_{\rm f}x)u(p_{\rm f},\lambda_{\rm f})\mathscr{A}_{\rm eff}[x|A_{\rm R},\mathscr{A}]\Psi[x;p_{\rm i},\lambda_{\rm i}]. \tag{32}$$

It can now be shown that in the low-frequency limit the scattering matrix is equal to

$$\begin{aligned}S_{\rm fi}=&-2\pi{\rm i}\sum_n m(E_{\rm i}E_{\rm f})^{-1/2}\delta(E_{\rm f}-E_{\rm i}-n\omega)\\&\times A_{0,n}\left(\frac{eap_{\rm i}}{kp_{\rm i}}-\frac{eap_{\rm f}}{kp_{\rm f}},\frac{e^2a^2}{8kp_{\rm i}}-\frac{e^2a^2}{8kp_{\rm f}}\right)\\&\times\int{\rm d}\bar{x}\,\bar{u}(p_{\rm f},\lambda_{\rm f})\exp(-{\rm i}\bar{p}_{\rm f}\bar{x})\mathscr{A}(\bar{x})\psi[\bar{x};p_{\rm i},\lambda_{\rm i}].\end{aligned} \tag{33}$$

This means that the cross section for n-photon absorption is of the form

$$\begin{aligned}\frac{{\rm d}\sigma^{(n)}}{{\rm d}\Omega}=&\frac{|\bar{p}_{\rm f}|}{|\bar{p}_{\rm i}|}\left|A_{0,n}\left(\frac{eap_{\rm i}}{kp_{\rm i}}-\frac{eap_{\rm f}}{kp_{\rm f}},\frac{e^2a^2}{8kp_{\rm i}}-\frac{e^2a^2}{8kp_{\rm f}}\right)\right|^2\\&\times\frac{{\rm d}\sigma}{{\rm d}\Omega}(\bar{p}_{\rm f},\bar{p}_{\rm i})|_{\text{no laser}}+{\rm O}\left(\frac{\omega}{E_{\rm i}-m}\right),\end{aligned} \tag{34}$$

which is the relativistic generalisation of the Kroll-Watson formula for an arbitrary *short-ranged* potential $\mathscr{A}(\bar{x})$. It is worth noting in this place that although the formula (34) was derived considering the electron scattering, it holds also for the scattering of charged mesons, hyperons, quarks, etc.

For almost all lasers available now the intensities and frequencies are such that the second argument of the generalised Bessel function $A_{0,n}$ is small. Hence, in these cases, the relativistic Kroll-Watson formula is of the form

$$\frac{{\rm d}\sigma^{(n)}}{{\rm d}\Omega}=\frac{|\bar{p}_{\rm f}|}{|\bar{p}_{\rm i}|}J_n^2\left(\frac{eap_{\rm i}}{kp_{\rm i}}-\frac{eap_{\rm f}}{kp_{\rm f}}\right)\frac{{\rm d}\sigma}{{\rm d}\Omega}(\bar{p}_{\rm f},\bar{p}_{\rm i})\bigg|_{\text{no laser}}+{\rm O}\left(\frac{\omega}{E_{\rm i}-m}\right), \tag{35}$$

which, for the non-relativistic particles, can be reduced to (1).

5. Conclusion and prospects

In this paper relativistic electron scattering within a laser field has been considered but the final result (formula (34)) is fulfilled for any charged particle. For sufficiently intense lasers (e.g., the CO_2 laser) and special configurations of the incident and scattered momenta of the particle and the polarisation of the laser beam the suppression of the cross section for n-photon absorption ($n > 0$) or emission ($n < 0$) becomes possible. Since the non-resonant background of the collision can be described in terms of the potential scattering, therefore, we can hope it is possible to extract the resonant part of the scattering process, as it was postulated in the case of the non-relativistic scattering by Jung and Taylor (1981)†. The spacetime dependence of the laser field can also be included applying methods from the non-relativistic theory, since in both cases the same Bessel functions appear in front of the cross section without the laser field.

Consideration of the relativistic theory is also interesting for the reason that there exists, contrary to the non-relativistic theory, the exactly soluble model, i.e., the massless two-dimensional quantum electrodynamics (see, e.g., Becher 1983)‡. It can be hoped that with the help of this model many conjectures and results can be tested leading to the better understanding of the phenomenon considered.

The non-relativistic Kroll-Watson formula takes into account also the linear term in ω, expressing the cross section in terms of appropriately shifted momenta of incident and scattered electrons. Therefore, it is interesting to ask whether it is possible to include the linear terms in ω also in the relativistic case and whether the momentum shift depends on the spin character of $\mathscr{A}(\bar{x})$.

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† The experiment that is aimed at checking this phenomenon is under construction at the University of Southern California (H S Taylor, private communication).

‡ There exists the exactly soluble model in the non-relativistic quantum mechanics but only for the monochromatic circularly polarised plane wave in the dipole approximation (Berson 1975).

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